PUTNAM PRACTICE SET 2

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Problem 1. Consider the two sequences $\{a_m\}_{m\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ defined by

$$a_1 = 3 \text{ and for each } m \geq 1, \text{ we have } a_{m+1} = 3^{a_m}$$

and

$$b_1 = 100 \text{ and for each } n \geq 1, \text{ we have } b_{n+1} = 100^{b_n}.$$

Find the smallest possible integer n such that $b_n > a_{2019}$.

Problem 2. Let $n > 1$ be an integer and let $a > 0$ be a real number. Let $x_1, \dots, x_n$ be nonnegative real numbers satisfying: $\sum_{i=1}^{n} x_i = a$. Find the maximum of $\sum_{i=1}^{n-1} x_i x_{i+1}$.

Problem 3. Let N be the number of integer solutions to the equation $x^3 - y^3 = z^5 - t^5$ with the property that $0 \leq x, y, z, t \leq 2019^{2019}$. Let M be the number of integer solutions to the equation $x^3 - y^3 = z^5 - t^5 + 1$ with the property that $0 \leq x, y, z, t \leq 2019^{2019}$. Prove that $N > M$.

Problem 4. Find all $n \in \mathbb{N}$ such that $2^8 + 2^{11} + 2^n$ is a perfect square.